ノルムに到達するティープリッツ及びハンケル作用素の構造 (作用素論の発展と諸問題)

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数理解析研究所講究録
1189 巻 2001 年 52-57

ノルムに到達するテイープリッツ及びハンケル作用素の構造

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Let μ be the normalized Lebesgue measure on the Borel sets of the unit circle in the complex plane $\mathbb{C}$. If $e_n(z) = z^n$ for $|z| = 1$ and $n = 0, \pm 1, \pm 2, \cdots$, then the bounded measurable functions e_n constitute an orthonormal basis for $L^2 = L^2(\mu)$. And the functions e_n, $n = 0, 1, 2, \cdots$ constitute an orthonormal basis for H^2. Let ${H_0}^2 = \{f \in H^2 : f(0) = 0\}$. L^∞ denotes the set of all essentially bounded measurable functions on the unit circle and $H^\infty = H^2 \cap L^\infty$. For a $\varphi \in L^\infty$ the Laurent operator L_φ is given by $L_\varphi f = \varphi f$ for $f \in L^2$ as the multiplication operator on L^2. And the Laurent operator induces, in a natural way, twin operators on H^2 called the Toeplitz operator T_φ given by $T_\varphi f = PL_\varphi f$ for $f \in H^2$ where P is the orthogonal projection from L^2 onto H^2 and the Hankel operator H_φ given by $H_\varphi f = J(I-P)L_\varphi f$ for $f \in H^2$ where J is the unitary operator on L^2 defined by $J(z^{-n}) = z^{n-1}$, $n = 0, \pm 1, \pm 2, \cdots$.

次の２つの結果は昨年の数理研の研究集会で示した（講究録 1137，108-111）．

Theorem 1. 次は同値である．

(1) $\{f \in H^2 : \|T_\varphi f\|_2 = \|T_\varphi\| \|f\|_2\} \neq \{o\}$ (i.e., T_φ is norm-achieved).

(2) $\frac{\varphi}{\|T_\varphi\|} = g$ for some $g \in L^\infty$ such that $|g| = 1$ a.e. and that $0 \in \sigma_p(H_g)$.

In this case, $\{f \in H^2 : \|T_\varphi f\|_2 = \|T_\varphi\| \|f\|_2\} = \mathcal{N}_{H_g}$ and it is invariant under T_z.

(3) $\frac{\varphi}{\|T_\varphi\|} = \overline{q}h$ for some inner functions q and h such that q and h have no common non-constant inner factor.

Theorem 2. 次は同値である．

(1) $\{f \in H^2 : \|H_\varphi f\|_2 = \|H_\varphi\| \|f\|_2\} \neq \{o\}$ (i.e., H_φ is norm-achieved).

(2) $\frac{\varphi}{\|H_\varphi\|} = g + \psi$ for some $\psi \in H^\infty$ and $g \in L^\infty$ such that $|g| = 1$ a.e. and that $0 \in \sigma_p(T_g)$.

In this case, $\{f \in H^2 \ : \ \|H_\varphi f\|_2 = \|H_\varphi\|\|f\|_2\} = \mathcal{N}_{T_g}$.

The following results are known.

Proposition 1. ([1]) Let $\mathcal{M}$ be an invariant subspace of L_z. Then, in the case where $L_z\mathcal{M} = \mathcal{M}$, there exists a characteristic function χ_E of some subset E of the unit circle such that $\mathcal{M} = L_{\chi_E} L^2$ and, in the case where $L_z\mathcal{M} \subset \mathcal{M}$, there exists a unitary Laurent operator L_g uniquely, except a constant multiple of absolute value one, such that $\mathcal{M} = L_g H^2$.

Lemma 1. ([2]) If φ is non-analytic (i.e., $\varphi \notin H^\infty$), then the only invariant subspace of L_φ which contained in H^2 is $\{o\}$ itself.

Lemma 2. If φ is non-co-analytic (i.e., $\overline{\varphi} \notin H^\infty$), then the only invariant subspace of L_φ which contained in $L^2 \ominus {H_0}^2$ is $\{o\}$ itself.

Proof. Let

$$\mathcal{M} = \vee\{{L_\varphi}^{*n} f \ : \ f \in {H_0}^2, \ n = 0, 1, 2, \cdots\}.$$

Then it is the smallest invariant subspace of ${L_\varphi}^*$ which includes ${H_0}^2$. Hence we have only to prove $\mathcal{M} = L^2$. Since L_z commutes with ${L_\varphi}^*$ and since ${H_0}^2$ is invariant under L_z, $\mathcal{M}$ is invariant under L_z.

If $\mathcal{M}$ reduces L_z, then $\overline{z}^{n-1} = {L_z}^{*n} z \in \mathcal{M}$ $(n = 1, 2, \cdots)$ because $z \in {H_0}^2 \subseteq \mathcal{M}$ and hence $\mathcal{M} = L^2$.

If $\mathcal{M}$ is a non-reducing invariant subspace of L_z, then $L_z\mathcal{M} \subset \mathcal{M}$ because L_z is unitary and, by Proposition 1, $\mathcal{M} = L_g H^2$ for some unitary Laurent operator L_g and $L_g {L_\varphi}^* H^2 = {L_\varphi}^* \mathcal{M} \subseteq \mathcal{M} = L_g H^2$ and hence ${L_\varphi}^* H^2 \subseteq H^2$. Since $1 \in H^2$, $\overline{\varphi} \in H^2$ and $\overline{\varphi} \in H^2 \cap L^\infty = H^\infty$. This contradicts the hypothesis that φ is non-co-analytic. $\square$

Theorem 3. ([2]) For a T_φ such as $\|T_\varphi\| = 1$, if

$$\{f \in H^2 \ : \ \|{T_\varphi}^n f\|_2 = \|f\|_2, \ n = 0, 1, 2, \cdots\} \neq \{o\},$$

then T_φ is an isometry.

Proof. For a non-zero $f \in \{f \in H^2 \ : \ \|T_\varphi{}^n f\|_2 = \|f\|_2,\ n = 0, 1, 2, \cdots\}$, we have $\|f\|_2 = \|PL_\varphi f\|_2 \le \|L_\varphi f\|_2 \le \|f\|_2$ because $\|L_\varphi\| = \|T_\varphi\| = 1$. This implies that $T_\varphi f = PL_\varphi f = L_\varphi f$ and

$$\|f\|_2 = \|T_\varphi{}^2 f\|_2 = \|T_\varphi L_\varphi f\|_2 = \|PL_\varphi{}^2 f\|_2 \le \|L_\varphi{}^2 f\|_2 \le \|f\|_2$$

and hence $T_\varphi{}^2 f = PL_\varphi{}^2 f = L_\varphi{}^2 f$. Similarly, we have $T_\varphi{}^n f = PL_\varphi{}^n f = L_\varphi{}^n f$ for all $n \ge 0$.

Let $\mathcal{N} = \vee\{L_\varphi{}^n f \ : \ n = 0, 1, 2, \cdots\}$. Then $\mathcal{N} \neq \{o\}$ is an invariant subspace of L_φ contained in H^2 and, by Lemma 1, φ is analytic, i.e., $\varphi \in H^\infty$. Since, by Theorem 1, $\varphi = \overline{q}h$ for some inner functions q and h such that q and h have no common non-constant inner factor, $h = q\varphi$ and $q = e^{i\theta_0}1$ for some $\theta_0 \in [0, 2\pi)$ and hence $\varphi = e^{-i\theta_0}h$ is inner. $\square$

Corollary 1. ([2]) For a non-constant function φ in L^∞, if T_φ is a contraction (i.e., $\|T_\varphi\| \le 1$), then it is completely non-unitary.

Proof. It is known that the unitary part of T_φ is the restriction of T_φ on

$$\mathcal{H}^{(u)}_{T_\varphi} = \{f \in H^2 \ : \ \|T_\varphi{}^n f\|_2 = \|T_\varphi{}^{*n} f\|_2 = \|f\|_2,\ n \ge 0\}.$$

Hence we have only to prove $\mathcal{H}^{(u)}_{T_\varphi} = \{o\}$. If $\mathcal{H}^{(u)}_{T_\varphi} \neq \{o\}$, then $\varphi \in H^\infty \cap \overline{H^\infty} = \{\mathbb{C}1\}$ by Theorem 3. $\square$

The matrix representation $(a_{i,j})_{i,j=0}^\infty$ of H_φ where $a_{i,j} = \langle H_\varphi z^j,\ z^i\rangle$ by the basis $\{z^n \ : \ n \ge 0\}$ is determined by $\{a_{i,0} \ : \ i \ge 0\}$ and we remark here $\sum_{i=0}^\infty a_{i,0} z^i = H_\varphi 1$. And hence we have the following.

Theorem 4. $H_\varphi = O$ if and only if $H_\varphi 1 = o$.

It is known that there is no invertible Hankel operator. Moreover, we have the following.

Corollary 2. There is no isometric Hankel operator.

Proof. If $H_\varphi{}^* H_\varphi = I$, then

$$H_\varphi{}^* H_\varphi = I = T_z{}^* T_z = T_z{}^* H_\varphi{}^* H_\varphi T_z = H_\varphi{}^* T_z T_z{}^* H_\varphi$$

and $H_\varphi{}^*(I-T_zT_z{}^*)H_\varphi = O$ and hence $(I-T_zT_z{}^*)H_\varphi = O$ because $I-T_zT_z{}^*$ is a projection. And then $H_\varphi H^2 \subseteq T_zH^2$ and $\langle H_\varphi{}^*1,\ H^2\rangle = \langle 1,\ H_\varphi H^2\rangle = 0$. Therefore $H_{\varphi^*}1 = H_\varphi{}^*1 = o$ and, by Theorem 4, $H_\varphi{}^* = H_{\varphi^*} = O$ which contradicts $H_\varphi{}^*H_\varphi = I$. $\square$

Theorem 5. For a H_φ such as $\|H_\varphi\| = 1$, if

$$\{f \in H^2\ :\ \|H_\varphi{}^nf\|_2 = \|f\|_2,\ n = 0,1,2,\cdots\} \neq \{o\},$$

then H_φ is normal.

Proof. Since, by Theorem 2, $\varphi = g+\psi$ for some $\psi \in H^\infty$ and $g \in L^\infty$ such as $|g| = 1$ a.e. and $0 \in \sigma_p(T_g)$. Hence $H_\varphi = H_g$.

For a non-zero $f \in \{f \in H^2\ :\ \|H_\varphi{}^nf\|_2 = \|f\|_2,\ n \geq 0\}$,

$$\|f\|_2 = \|J(I-P)L_gf\|_2 \leq \|f\|_2$$

because $\|L_g\| = \|g\|_\infty$ and $(I-P)L_gf = L_gf$ and hence $H_gf = JL_gf$. Since

$$\|f\|_2 = \|H_g{}^2f\|_2 = \|(I-P)L_gJL_gf\|_2 \leq \|f\|_2,$$

$(I-P)L_gJL_gf = L_gJL_gf = JL_{\overline{g^*}}L_gf$ and $H_g{}^2f = L_{\overline{g^*}}L_gf$. And

$$\|f\|_2 = \|H_g{}^3f\|_2 = \|(I-P)L_gL_{\overline{g^*}}L_gf\|_2 \leq \|f\|_2.$$

Hence $(I-P)L_gL_{\overline{g^*}}L_gf = L_gL_{\overline{g^*}}L_gf$ and $H_g{}^3f = JL_gL_{\overline{g^*}}L_gf$. Similarly, for all $n \geq 0$,

$$\begin{aligned} & (I-P)(L_gL_{\overline{g^*}})^nL_gf = (L_gL_{\overline{g^*}})^nL_gf \\ \text{and}\qquad & (I-P)J(L_{\overline{g^*}}L_g)^nf = J(L_{\overline{g^*}}L_g)^nf. \end{aligned}$$

Let $\mathcal{N} = \vee\{L_{g\overline{g^*}}{}^nL_gf\ :\ n \geq 0\}$. Then $\mathcal{N} \neq \{o\}$ is an invariant subspace of $L_{g\overline{g^*}}$ contained in $L^2 \ominus H_0{}^2$ and, by Lemma 2, $g\overline{g^*}$ is co-analytic and hence $u = \overline{g}g^*$ is inner because $|\overline{g}g^*| = |g|\ |g^*| = 1$ a.e. Since $u^*u = g\overline{g^*}\overline{g}g^* = 1$, $\overline{u} = u^* \in H^\infty$ and hence u is a constant of absolute value one because $u \in H^\infty \cap \overline{H^\infty} = \mathbb{C}1$. Therefore $g^* = e^{i\theta_0}g$ for some $\theta_0 \in [0,2\pi)$ and we have the conclusion. $\square$

Corollary 3. If H_φ is a non-normal contraction, then it is completely non-unitary.

Proof. We have only to prove that $\mathcal{H}_{H_\varphi}^{(u)} = \{o\}$. If $\mathcal{H}_{H_\varphi}^{(u)} \neq \{o\}$, then H_φ is normal by Theorem 5. $\square$

We say that a bounded linear operator A on a Hilbert space $\mathcal{H}$ is paranormal if $\|Ax\|^2 \leq \|A^2x\| \ \|x\|$ for all $x \in \mathcal{H}$.

Theorem 6. If T_φ is norm-achieved paranormal, then T_φ is a scalar multiple of an isometry.

Proof. We may assume that $\|T_\varphi\| = 1$. Let

$$\mathcal{M} = \{f \in H^2 : \|T_\varphi f\|_2 = \|f\|_2\}.$$

Then , by the hypothesis, $\mathcal{M} \neq \{o\}$ and $T_\varphi \mathcal{M} \subseteq \mathcal{M}$ by the paranormality of T_φ. In fact, if $f \in \mathcal{M}$, then we have

$$\|f\|_2{}^2 \geq \|f\|_2 \ \|T_\varphi{}^2 f\|_2 \geq \|T_\varphi f\|_2{}^2 = \|f\|_2 \ \|T_\varphi f\|_2 = \|f\|_2{}^2$$

and this implies that $\|T_\varphi{}^2 f\|_2 = \|T_\varphi f\|_2$ and hence $T_\varphi f \in \mathcal{M}$. Therefore

$$\{f \in H^2 \ : \ \|T_\varphi{}^n f\|_2 = \|f\|_2, \ n = 0, 1, 2, \cdots\} \neq \{o\}$$

and T_φ is an isometry by Theorem 3. $\square$

Theorem 7. If H_φ is norm-achieved paranormal, then H_φ is normal.

Proof. We may assume that $\|H_\varphi\| = 1$. Let

$$\mathcal{M} = \{f \in H^2 : \|H_\varphi f\|_2 = \|f\|_2\}.$$

Then , by the hypothesis, $\mathcal{M} \neq \{o\}$ and, by the paranormality of H_φ, $H_\varphi \mathcal{M} \subseteq \mathcal{M}$. In fact, if $f \in \mathcal{M}$, then we have

$$\|f\|_2{}^2 \geq \|f\|_2 \ \|H_\varphi{}^2 f\|_2 \geq \|H_\varphi f\|_2{}^2 = \|f\|_2 \ \|H_\varphi f\|_2 = \|f\|_2{}^2$$

and $\|H_\varphi{}^2 f\|_2 = \|H_\varphi f\|_2$ and hence $H_\varphi f \in \mathcal{M}$. Therefore

$$\{f \in H^2 \ : \ \|H_\varphi{}^n f\|_2 = \|f\|_2, \ n = 0, 1, 2, \cdots\} \neq \{o\}$$

and H_φ is normal by Theorem 5. $\square$

References

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